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A Method to Determine the Critical Parameter Tolerances for a Proportional-Integral-Derivative (PID) Controller

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The PID controllers are one of the most important control elements used in process control industry. In this study, a general method is proposed to determine the critical parameter tolerances by the use of the sensitivities of a PID controller's transfer function. These tolerances keep the relative error at the output of the PID controller due to parameter variations in a predefined tolerance region.

Keywords: Sensitivity definitions, proportional-integral-derivative (PID) controller, critical parameter tolerances.

1. Introduction

Over the last 15 years, control theory has developed several techniques for designing linear time-invariant control systems that are optimum and robust. It is shown that optimum and robust controllers designed by these techniques can produce extremely fragile controllers, in the sense that vanishingly small deviations of the coefficients due to the environmental effects of the controller designed destabilize the closed-loop system. The fragility also shows up usually as extremely small gain and/or phase margins of the closed-loop system [1].

It is obvious that it would be unwise to place a controller that is fragile with respect to deviations of its coefficients due to the environmental effects in an actual control system without further precautions and analysis since deviations of controller coefficients cause the deviation of the output voltage of the controller.

Although reducing the deviations at the output due to the environmental effects are studied extensively [2,3,4,5,6] in active network design, this problem has not been examined and solved so far for controllers. Considering this gap in the literature, the following a method is proposed to solve this problem for PID controller by the use of the sensitivity concept. For this purpose, using the transfer function of a PID controller and the parameter sensitivities, first an upper bound is given for the deviation at the output voltage of the PID controller. Then the critical parameter tolerances which satisfy this upper bound are presented. In this way, the total error of a closed-loop control system, which is equal to the sum of the errors of the individual blocks, can be satisfied more easily.

2. Basic Definitions

The relation between input and the output voltages of a PID controller can be written in s-domain as

$$E_o(s) = T(s)E_i(s), \tag{1}$$

where $e_o(t)$ is the output voltage; $e_i(t)$ is the input voltage. In Eq.1, T(s) is the transfer function of a general analog, PID controller, which can be written as [7,8,9,10]

$$T(s) = \frac{E_o(s)}{E_i(s)} = K_1 + \frac{K_2}{s} + sK_3, \tag{2}$$

where K_1 is the proportional gain; K_2 is the integral gain, and K_3 is the derivative gain.

The relative deviation of the output voltage can be expressed in terms of parameter sensitivities, as follows [11,12,13]:

$$\frac{\Delta E_o(s)}{E_o(s)} = \frac{\Delta T(s)}{T(s)} = \sum_{k=1}^{n} S_{x_k}^{T}(s)(\frac{\Delta x_k}{x_k}), \tag{3}$$

where x_k, $k = 1, ..., n$ denotes the nominal value of k^{th} parameter which is either a component or a system parameter and $\Delta x_k / x_k$ is the relative deviation in kth parameter due to the environmental effects, and $S_{x_k}^{T}(s)$ is the normalized sensitivity of the transfer function, T(s) with respect to kth parameter x_k and is defined as:

$$S_{x_k}^T(s) = (\frac{\partial T}{T})/(\frac{\partial x_k}{x_k}), \quad k = 1, ..., n. \tag{4}$$

The sensitivity, $S_{x_k}^T(s)$ can also be written in terms of gain and phase sensitivities after substituting $s = j\omega$, as follows [5]:

$$S_{x_k}^T(j\omega) = S_{x_k}^{|T|}(\omega) + jS_{x_k}^{\beta}(\omega), \tag{5}$$

where $S_{x_k}^{|T|}(\omega)$ and $S_{x_k}^{\beta}(\omega)$ are normalized sensitivity of the gain function and semi-normalized sensitivity of the phase function respectively and can be written by using the chain rule as

$$S_{x_k}^{|T|}(\omega) = \sum_{m=1}^{3} \frac{x_k}{|T|} \frac{\partial |T|}{\partial K_m} \frac{\partial K_m}{\partial x_k}, \tag{6}$$

$$S_{x_k}^{\beta}(\omega) = \sum_{m=1}^{3} x_k \frac{\partial \beta}{\partial K_m} \frac{\partial K_m}{\partial x_k}. \tag{7}$$

The overall relative deviation at the output voltage of a controller due to parameter variations can be obtained from Eq. 3, Eq. 6, and Eq. 7 as

$$\frac{\Delta E_o(j\omega)}{E_o(j\omega)} = \sum_{k=1}^{n} [S_{x_k}^{|T|}(\omega) + jS_{x_k}^{\beta}(\omega)] \frac{\Delta x_k}{x_k}. \tag{8}$$

With this formula, a designer can evaluate the relative deviation at the output of a PID controller, once parameter variations are known.

3. Calculation Of Critical Parameter Tolerances

In practice, it is very difficult to predict the parameter variations but their upper bounds named as their tolerances. Hence it is quite hard to estimate the output deviation. Therefore, in the following, first an upper bound will be given for $\Delta E_o/E_o$, and then using this upper bound critical parameter tolerances will be calculated.

By the use of the triangular inequality together with Eq. 8, the upper bound for overall relative deviation at the output voltage can be expressed as follows, for the designer's specified frequency band of $\omega \in [\omega_1, \omega_2]$ [14]:

$$|\frac{\Delta E_o(j\omega)}{E_o(j\omega)}| \leq \sum_{k=1}^{n} \{\sqrt{[S_{x_k}^{|T|}(\omega)]^2 + [S_{x_k}^{\beta}(\omega)]^2}\} t_k$$

$$\leq \sum_{k=1}^{n} \{\sqrt{[S_{x_k}^{|T|}(\omega)]^2 + [S_{x_k}^{\beta}(\omega)]^2}\} t_{cr} \leq t_o, \tag{9}$$

where t_k is the tolerance of kth parameter defined as,

$$\max |\frac{\Delta x_k}{x_k}| = t_k, \tag{10}$$

and t_{cr}, is the largest tolerance among t_ks and is named as the critical parameter tolerance:

$$t_{cr} = \max\{t_k, \quad k = 1, ..., n\}. \tag{11}$$

Furthermore, t_o is the output tolerance defined as the maximum value of $\sum_{k=1}^{n} |S_{x_k}^T(j\omega)|$ versus ω, multiplied by t_{cr}:

$$t_o = t_{cr}[\sum_{k=1}^{n} |S_{x_k}^T(j\omega)|]_{\max}, \ \omega \in [\omega_1, \omega_2]. \tag{12}$$

Note that t_o is the upper bound for $|\Delta E_o/E_o|$. Eq. 12 can be used in calculating the value of the critical parameter tolerance keeping $|\Delta E_o/E_o| \leq t_o$ in a specified frequency band.

By assigning ω_c as the critical angular frequency at which $\sum_{k=1}^{n} |S_{x_k}^T(\omega)|$ takes its maximum value, we can calculate the critical parameter tolerance as

$$t_{cr} = \frac{t_o}{\sum_{k=1}^{n} |S_{x_k}^T(\omega_c)|}. \tag{13}$$

Note from Eq. 13 that if the designer chooses each of the parameter tolerances less than or equal to t_{cr} the relative deviation at the output of the controller falls always in prescribed tolerance region denoted by t_o.

In the following, the critical parameter tolerances will be calculated according to the proposed formula for an electronic PID controller in Figure 1 [15,16]. The controller gains can be written as

$$K_1 = R_4(R_1C_1 + R_2C_2)/R_1R_3C_2, \tag{14a}$$

$$K_2 = R_4/R_1R_3C_2, \tag{14b}$$

$$K_3 = R_2R_4C_1/R_3. \tag{14c}$$

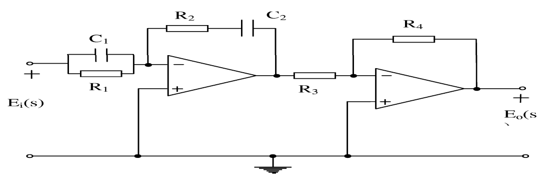

Figure 1. An Electronic, Proportional-Integral-Derivative (PID) Controller

Let the controller gains be specified as $K_1 = 39.62$, $K_2 = 12.87s^{-1}$, and $K_3 = 30.48s$. Then the parameter values can be chosen as follows:

$$R_1 = R_2 = 153.85K\Omega, \quad R_3 = 10K\Omega, \tag{15a}$$

$$R_4 = 197.1K\Omega, \quad C_1 = C_2 = 10\mu F. \tag{15b}$$

Assuming that the designer wants $|\Delta E_o/E_o|$ to be less than or equal to 0.01, then the critical parameter tolerance is to be obtained as

$$t_{cr} = 0.2\%. \tag{16}$$

Therefore, the parameter tolerances must be chosen smaller than the critical parameter tolerance. If the designer wants $|\Delta E_o/E_o|$ to be less than or equal to 0.1, then the critical parameter tolerances may be chosen ten times larger than the ones in Eq.16, and so forth.

4. Conclusions

In this study, using the sensitivities of the transfer function of a PID controller, a general method is proposed to determine the critical parameter tolerances for any kinds of PID controllers by an appropriate approach. If the parameter tolerances are chosen less than or equal to the critical parameter tolerance, t_{cr}, the relative error at the output of a PID controller due to the parameter variations always stays in prescribed tolerance region denoted by t_o. In this way, the total error of a closed-loop control system, which is equal to the sum of the errors of the individual blocks, can easily be satisfied.

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