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Transfer and Stiffnes Matrix for Timoshenko Beams on Elastic Foundations

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In this study Timoshenko beams on elastic foundations are analyzed under arbitrary loading by transfer matrix method. The Winkler hypothesis adopted, because of its simplicity. The analytical approaches provide physical insight into the nature of the problem. It is observed that three distinct behavior of a structure of the beam exist depending on parameters. Therefore three types of transfer matrixes are derived by the spectral expansion method, in terms of the displacement, the rotation, the moment and the shear forces. Furthermore, three distinct stiffness matrixes are obtained. However, for engineering purposes only one of them has practical importance. The transfer matrix serves to derive the stiffness matrix, which is necessary to analyze the structural frames. For this purpose equivalent nodal forces are given for the concentrated and trapezoidal load distribution. The performance of the method is shown by the numerical examples.

Keywords: Timoshenko beam, elastic foundation, transfer matrix, stiffness matrix of Timoshenko beam

1. Introduction

It is well known from the classical beam theory that, shear effect is important for the beams with small span-to- height ratio or beams which have profile cross sections or in other words for the beams which have a high cross section parameter. In addition, shear effect may be important for beams which are subjected to concentrated loads. A beam theory that considers for both the bending and shearing was proposed by Timoshenko for modeling behaviors of the beams [1]. It is commonly accepted that Timoshenko beam theory is more accurate than the Bernoulli-Euler theory, particularly when the beam has the properties which are stated previously. In the other hand, the response of foundations, the Winkler hypothesis is extensively used by researchers because of its simplicity. A summary of foundation models introduced by numerous investigators is given in the reference [2]. Beams on elastic foundations have been investigated by numerous researchers. Hetenyi [3] studied straight beams on Winkler foundations and obtained an exact solution. Various authors [4, 5] have investigated closed form solutions of the problem, beams on elastic foundation. Keskinel have derived the stiffness matrix for the beams on elastic foundation [6]. The finite element approaches employed extensively in the analysis of the structures. The general information can be found in literature [7, 8].

Aköz et al [9] introduce a mixed finite element formulation for three-dimensional beams which accounts for the effect of shear deformation and was shown that the exact result can be obtained with a single element for the cantilever beam using this formulation. Friedman and Kosmate [10] developed an accurate two node finite element for curved shear deformable beams. A least squares finite element method for Timoshenko beam problem has been proposed and studied by Jou and Young [11]. Aköz and Kadıoğlu [12] have studied circular Timoshenko beam resting on elastic foundation by finite element method. Aydoğan obtained numerical stiffness matrix of beam element with shear effect, using the shape functions which are solutions of the fourth order differential equations [13].

Another approach in solving the differential equations is matrix method that originally introduced by Inan [14] and it named as carry over-matrix. In subsequent years, other papers have appeared in literature [15-18]. Yin derived ordinary differential equations for reinforced Timoshenko beams on elastic foundation. An analytical solution is obtained for a point load on the infinite beam on elastic foundation [19]. Yin later, obtained the closed-form solution of finite beam under any vertical pressure, using Fourier series [20]. The integral equation method is used by Antes. He chose displacement and rotation as free parameters [21]. Alghamdi derived the dynamic

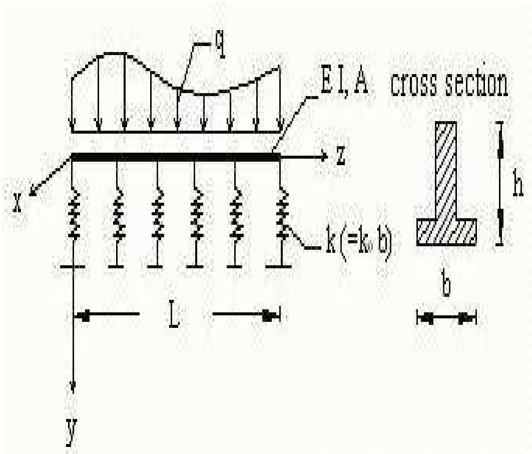

Figure 1. The Timoshenko beam on elastic foundation

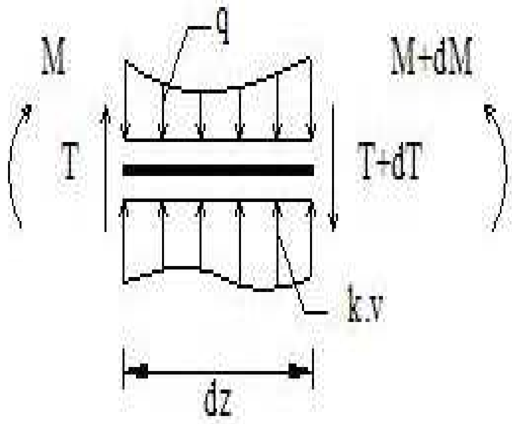

Figure 2. an infinitesimal element and positive direction of internal forces

transport matrix for the Timoshenko beam, in terms of two independent parameters i.e. displacement and rotation of the beam [22].

In this study, we shall deal with the solution of the system of equations employing the transfer matrix method. Timoshenko beam on elastic foundation displays three distinct behaviors depending on the material properties. Three different transfer matrix are obtained by spectral expansion method in terms of four parameters i.e. the displacement v, the rotation of the beam, θ, the bending moment M and the shear force T. In the second step, stiffness matrixes are derived to implement computer programs for frame analysis.

2. The Field Equations

The beam on elastic foundation and the axes are depicted in Fig. 1. Longitudinal axis is denoted as z in this figure. Where EI is the bending rigidity, A is the beam cross section area, L is the beam length, k is the sub grade reaction modulus, q is the lateral distributed load, b is wide and h is the height of beam respectively. The sign convention for shearing force and bending moment depicted in Fig. 2. v shows the linear displacement in y direction.

Kinematics relation can be written as follows:

$$\frac{dv}{dz} = -\Omega + \frac{sT}{GA} \tag{1}$$

where Ω is the rotation of the cross section of the beam, s is the shape factor and defined as follows:

$$s = \frac{1}{A}\int_A\int\left(\frac{\tau}{\tau_0}\right)^2 dA \tag{2}$$

Employing Bernoulli-Navier Hypothesis, the Hooke Love and two equilibrium equations, we end up with the following three equations.

$$\begin{aligned} \frac{d\Omega}{dz} &= \frac{M}{EI} \\ \frac{dM}{dz} &= T \\ \frac{dT}{dz} &= -q + kv \end{aligned} \tag{3}$$

Combination of Eqs. (1) and (3) can be expressed by a single matrix equation as follow:

$$\mathbf{S}'(z) = \mathbf{D}\cdot\mathbf{S}(z)\,, \tag{4}$$

where **S** is called a state vector and contains the four parameters v, Ω, M, T. The positive direction of the elements state vector are depicted in Fig. 3.

And **D** is called the *differential transmitter matrix* (DTM) and defined as follow:

$$\mathbf{D} = \begin{bmatrix} 0 & -1 & 0 & \frac{s}{GA} \\ 0 & 0 & \frac{1}{EI} & 0 \\ 0 & 0 & 0 & 1 \\ k & 0 & 0 & 0 \end{bmatrix} \tag{5}$$

For more information the reader may refer to M. İnan [14, 17].

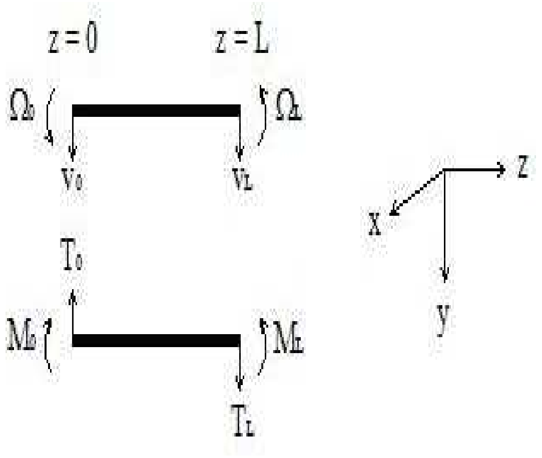

Figure 3. the positive direction of the element of the state vector

3. The Solution and Transfer Matrix

The solution of the Eq. (3) can be easily written as

$$\mathbf{S} = e^{x\mathbf{D}}\mathbf{S_0}\,, \tag{6}$$

where $\mathbf{S_0}$ is the initial value of the state vector at z=0. Then the transfer function can be defined as

$$\mathbf{F} = e^{x\mathbf{D}} \tag{7}$$

Eq. (6) can be written shortly as follows

$$\mathbf{S}(x) = \mathbf{F}(x)\mathbf{S}(0) \tag{8}$$

The transfer matrix $\mathbf{F}(x)$ will be written in terms of nilpotent and idempotent matrix. This is achieved by using spectral expansion theorem as follows [23]:

$$\mathbf{D} = \sum_{i=1}^{n} \lambda_i \mathbf{P}_i + \mathbf{Q}_i\,, \tag{9}$$

where, λ_i are eigenvalues of the matrix $\mathbf{D}$, $\mathbf{P}$ and $\mathbf{Q}$ are idempotent and nilpotent of $\mathbf{D}$ respectively. For the sake of simplicity the details of mathematical operation is not given here. The derivation and the properties of these matrixes are omitted. Necessary information can be found in literature [23]. If the idempotent and nilpotent of any matrix are known then the function of this matrix is written as

$$\Phi(\mathbf{D}) = \sum_{i=1}^{n} \left\{ \Phi(\lambda_i)\mathbf{P}_i + \sum_{1}^{s_k-1} \frac{\Phi^r(\lambda_i)}{r!}\mathbf{Q}_i^r \right\}\,, \tag{10}$$

where s_k is multiplicity of, λ_i, $r = s_k - 1$ and Φ^r is r^{th} degree of derivative with respect to λ_i. After this definition, the transfer function $\mathbf{F} = e^{z\mathbf{D}}$ can be derived using Eq. (9). Obviously to accomplish the defined mathematical operation, the first step is to calculate eigenvalues of matrix $\mathbf{D}$. Which are the roots of characteristic equation of $\mathbf{D}$:

$$\lambda^4 - \frac{sk}{GA}\lambda^2 + \frac{k}{EI} = 0\,. \tag{11}$$

The roots or eigenvalues are

$$\lambda_{1,2}^2 = \frac{sk}{2GA}\left(1 \pm \sqrt{1-\alpha}\right)\,, \tag{12}$$

where α is

$$\alpha = \frac{4G^2A^2}{s^2kEI}\,. \tag{13}$$

Dimensionless α defines three intervals:

$$\begin{aligned} \alpha &> 1 \\ \alpha &= 1 \\ \alpha &< 1\,. \end{aligned} \tag{14}$$

Referring to Eq. (13), it can be concluded that α must be grater than 1 for common engineering materials and the reasonable dimension of structures. Therefore, the first interval is important for engineering purposes. Then the basic procedure of mathematical operations will be explained and stiffness matrix will be developed only for this case. For the sake of completeness, only the transfer matrix will be given for the second and third intervals.

3.1. Transfer Matrix for $\alpha > 1$

For this case eigenvalues are

$$\lambda_{1,2}^2 = \left(\frac{sk}{2GA}\sqrt{1+\beta^2}\right)e^{\pm i\theta}\,, \tag{15}$$

where $\beta = \sqrt{\alpha - 1}$ and $\beta = \tan\theta$.

There are four idempotents. They are given in Appendix. Using Eq. (9) and idempotent the following transfer matrix is obtained.

$$\begin{aligned} F(z) = e^{zD} &= e^{z\lambda_1}P_1 + e^{z\lambda_2}P_2 \\ &\quad + e^{z\lambda_3}P_3 + e^{z\lambda_4}P_4\,. \end{aligned} \tag{16}$$

This is a fourth order matrix. The elements of the matrix are as follows:

$$F_{12} = L \cdot \Psi_2 \left(\Psi_3 \cdot \Psi_4 + \Psi_5 \cdot \Psi_6\right)$$

$$
\begin{aligned}
F_{13} &= -\frac{1}{E \cdot L^2}\left(\frac{\Psi_7 \cdot \sin\phi \sinh\varphi}{\sin 2\omega}\right) \\
F_{14} &= \frac{1}{2 \cdot E \cdot L}\{\Psi_8 \cdot [(1-2\cos 2\omega)\cdot \Psi_3 \\
&\quad + (1+2\cos 2\omega)\cdot \Psi_5] \\
&\quad + \Psi_9 \cdot [\Psi_3 - \Psi_5]\} \\
F_{21} &= \frac{1}{L}\left(\frac{\Psi_3 - \Psi_5}{4 \cdot \Psi_2}\right) \\
F_{22} &= \cos\phi \cdot \cosh\varphi - \frac{\sin\phi \cdot \cos 2\omega \cdot \sinh\varphi}{\sin 2\omega} \\
F_{23} &= -\frac{\Psi_{10}}{E \cdot L^3}\{\Psi_3 \cdot (-1 + 2\cos 2\omega) \\
&\quad + \cos\phi \cdot \sinh\varphi\,(\sec\omega - 4\cos\omega)\} \\
F_{24} &= \frac{1}{E \cdot L^2}\left(\frac{\Psi_7 \cdot \sin\phi \sinh\varphi}{\sin 2\omega}\right) \\
F_{31} &= E \cdot L^2\left(\frac{\sin\phi \sinh\varphi}{\Psi_7 \cdot \sin 2\omega}\right) \\
F_{32} &= E \cdot L^3\left(\frac{\Psi_5 - \Psi_3}{4 \cdot \Psi_{10}}\right) \\
F_{33} &= F_{22} \\
F_{34} &= -F_{12} \\
F_{41} &= k \cdot F_{34} \\
F_{42} &= -F_{31} \\
F_{43} &= -F_{21} \\
F_{44} &= F_{11}
\end{aligned} \tag{17}
$$

where

$$
\begin{aligned}
\mu &= 2s(1+v) \\
D^* &= EA \\
R^* &= \frac{1}{r^2} = \frac{A}{I} \\
\theta &= 2\omega \\
\phi &= z\left(\frac{kR^*}{D^*}\right)^{1/4} \sin\omega \\
\varphi &= z\left(\frac{kR^*}{D^*}\right)^{1/4} \cos\omega \\
a &= \left(\frac{k}{E}\right)^{1/4} \\
\Psi_1 &= \mu\left(a^2\right)\sqrt{\frac{I}{A^2}} \\
\Psi_2 &= \frac{1}{2a}\left(\frac{I}{L^4}\right)^{1/4} \\
\Psi_3 &= \sin\phi \csc\omega \cosh\varphi \\
\Psi_4 &= -\Psi_1 - 1 + 2\cos 2\omega \\
\Psi_5 &= \cos\phi \sec\omega \sinh\varphi \\
\Psi_6 &= \Psi_1 - 1 - 2\cos 2\omega \\
\Psi_7 &= \frac{1}{a^2}\sqrt{\frac{L^4}{I}} \\
\Psi_8 &= \frac{\mu}{a}\left(\frac{I \cdot L^4}{A^4}\right)^{1/4} \\
\Psi_9 &= a \cdot \mu^2\left(\frac{I^3 \cdot L^4}{A^8}\right)^{1/4} - \frac{1}{a^3}\left(\frac{L^4}{I}\right)^{1/4} \\
\Psi_{10} &= \frac{1}{2a}\left(\frac{L^{12}}{I^3}\right)^{1/4}
\end{aligned} \tag{18}
$$

3.2. Transfer Matrix for $\alpha < 1$

The detail of mathematical manipulation is omitted for this case and only the fourth order transfer matrix is given as:

$$
\begin{aligned}
F_{11} &= \frac{1}{2}\{\cosh\phi^* + \cosh\varphi^* \\
&\quad + \frac{(-\cosh\phi^* + \cosh\varphi^*)}{\sqrt{1-\alpha}}\} \\
F_{12} &= L \cdot Y_1\left(Y_2 \cdot \sinh\phi^* - Y_3 \cdot \sinh\varphi^*\right) \\
F_{13} &= \frac{A}{I \cdot k}\left(\frac{\cosh\phi^* - \cosh\varphi^*}{\mu\sqrt{1-\alpha}}\right) \\
F_{14} &= \frac{Y_4}{E \cdot L}\left(Y_5 \cdot \sinh\phi^* + Y_6 \cdot \sinh\varphi^*\right) \\
F_{21} &= \frac{Y_7}{L}\left(-\frac{\sinh\phi^*}{\sqrt{1-\sqrt{1-\alpha}}}\right. \\
&\quad \left. + \frac{\sinh\varphi^*}{\sqrt{1+\sqrt{1-\alpha}}}\right) \\
F_{22} &= \frac{1}{2}\{\cosh\phi^* + \cosh\varphi^* \\
&\quad + \frac{(\cosh\phi^* - \cosh\varphi^*)}{\sqrt{1-\alpha}}\} \\
F_{23} &= \frac{Y_8}{E \cdot L^3}\left(Y_9 \cdot \sinh\phi^* - Y_{10} \cdot \sinh\varphi^*\right) \\
F_{24} &= -\frac{A}{I \cdot k}\left(\frac{\cosh\phi^* - \cosh\varphi^*}{\mu\sqrt{1-\alpha}}\right) \\
F_{31} &= EA\left(\frac{-\cosh\phi^* + \cosh\varphi^*}{\mu\sqrt{1-\alpha}}\right) \\
F_{33} &= F_{22} \\
F_{34} &= -F_{12} \\
F_{41} &= k \cdot F_{34} \\
F_{42} &= -F_{31} \\
F_{43} &= -F_{21} \\
F_{44} &= F_{11}
\end{aligned} \tag{19}
$$

where

$$
\begin{aligned}
\phi^* &= z\left(\sqrt{\frac{\mu k-\sqrt{km}}{2D^*}}\right)\\
\varphi^* &= z\left(\sqrt{\frac{\mu k+\sqrt{km}}{2D^*}}\right)\\
Y_1 &= \sqrt{\frac{A}{L^2}}\\
Y_2 &= \frac{\sqrt{1-\sqrt{1-\alpha}}}{\sqrt{2a^4\mu\left(1-\alpha\right)}}\\
Y_3 &= \frac{\sqrt{1+\sqrt{1-\alpha}}}{\sqrt{2a^4\mu\left(1-\alpha\right)}}\\
Y_4 &= \frac{1}{2}\sqrt{\frac{L^2\mu}{2A\left(1-\alpha\right)a^4}}\\
Y_5 &= \frac{\left\{2\left(-1+\sqrt{1-\alpha}\right)+\alpha\right\}}{\sqrt{1-\sqrt{1-\alpha}}}\\
Y_6 &= \frac{\left\{2\left(1+\sqrt{1-\alpha}\right)-\alpha\right\}}{\sqrt{1+\sqrt{1-\alpha}}}\\
Y_7 &= \sqrt{\frac{2A^3L^2}{I^2a^4\mu^3\left(1-\alpha\right)}}\\
Y_8 &= \sqrt{\frac{AL^6}{2I^2a^4\mu\left(1-\alpha\right)}}\\
Y_9 &= \frac{\left(1+\sqrt{1-\alpha}\right)^{3/2}}{\sqrt{\alpha}}\\
Y_{10} &= \frac{\left(1-\sqrt{1-\alpha}\right)^{3/2}}{\sqrt{\alpha}}\\
Y_{11} &= \sqrt{\frac{2A^3}{L^6a^4\mu^3\left(1-\alpha\right)}}
\end{aligned}
\tag{20}
$$

3.3. The Transfer Matrix for $\alpha = 1$

The elements of fourth order transfer matrix reduces to the following simple form

$$
\begin{aligned}
F_{11} &= \cosh z\lambda+\frac{z\lambda}{2}\sinh z\lambda\\
F_{12} &= -\frac{z\lambda\cosh z\lambda+\sinh z\lambda}{2\lambda}\\
F_{13} &= -\frac{z\lambda^3}{2k}\sinh z\lambda\\
F_{14} &= \frac{\lambda}{2k}\left(z\lambda\cosh z\lambda+3\sinh z\lambda\right)\\
F_{21} &= \frac{\lambda}{2}\left(z\lambda\cosh z\lambda-\sinh z\lambda\right)\\
F_{22} &= \cosh z\lambda-\frac{z\lambda}{2}\sinh z\lambda\\
F_{23} &= -\frac{\lambda^3}{2k}\left(z\lambda\cosh z\lambda-3\sinh z\lambda\right)\\
F_{24} &= \frac{z\lambda^3}{2k}\sinh z\lambda\\
F_{31} &= \frac{kz}{2\lambda}\sinh z\lambda\\
F_{32} &= -\frac{k}{2\lambda^3}\left(-z\lambda\cosh z\lambda+\sinh z\lambda\right)\\
F_{33} &= F_{22}\\
F_{34} &= -F_{12}\\
F_{41} &= kF_{34}\\
F_{42} &= -F_{31}\\
F_{43} &= -F_{21}\\
F_{44} &= F_{11}
\end{aligned}
\tag{21}
$$

where

$$
\begin{aligned}
\lambda_1 &= \lambda=\sqrt{\frac{\mu k}{2D^*}}\\
D^* &= EA
\end{aligned}
\tag{22}
$$

4. The Properties of Transfer Matrix

A state vector of the beam at arbitrary point z has four elements which are the displacement v(z), the rotation $\Omega(z)$, the bending moment M(z) and the shear force T(z). As we move from one point to another along the axis of the beam, the elements of state vector varies in magnitude. The convenient way to express the change in S(z) is employment of the transfer matrix as follow:

$$\mathbf{S}(z)=\mathbf{F}(z)\cdot\mathbf{S}(0)\,, \tag{23}$$

where $\mathbf{S}(0)$ is called the initial state vector at z=0. This equation is very helpful in the study of Timoshenko beams on a elastic foundation. Using this equation the state vectors of two chosen specific points of the beam can be related to each other as follows:

$$\mathbf{S}(B)=\mathbf{F}(L)\mathbf{S}(A)\,, \tag{24}$$

where L is the distance between A and B. The points A and B are the support points of the beam. For any kind of support, two elements of state vector belonging to the supports of the beam, must be prescribed. Therefore, Eq. (24) is fourth order simultaneous equation for the unknown elements of state vectors. There are two unknowns on both sides. Therefore we can find

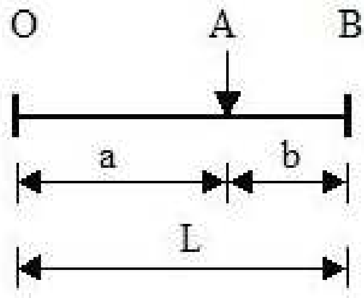

Figure 4. The singularity at point A

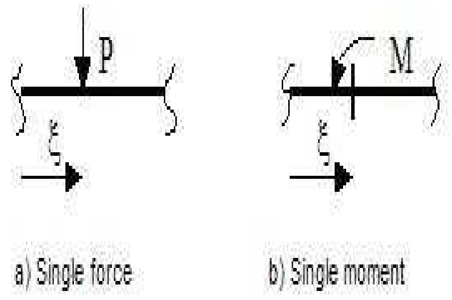

Figure 5. The different type of load

the results by solving second order equations. This property provides significant simplicity to the solution of the complex problem. The validity of the following properties can be shown by mathematical calculation which will be helpful [14]:

Property I

The transfer matrix $\mathbf{F}$(z) reduces to identity matrix $\mathbf{I}$ at z=0. That is

$$\mathbf{F}(0) = \mathbf{I} \tag{25}$$

Property II

The inverse of the transfer matrix can be obtained easily as follows:

$$\mathbf{F}^{-1}(a) = \mathbf{F}(-a) \tag{26}$$

Property III

$$\mathbf{F}(b) \cdot \mathbf{F}(a) = \mathbf{F}(a+b) \tag{27}$$

The property III, comes from the exponential character of the transfer matrix Eq. (7). Using this property we can reach to the point B directly or via the point A as follows (Fig. 4):

$$\begin{aligned} \mathbf{S}(B) &= \mathbf{F}(a+b) \cdot \mathbf{S}(0) \\ \mathbf{S}(B) &= \mathbf{F}(b) \cdot \mathbf{F}(a)\mathbf{S}(0) \end{aligned} \tag{28}$$

The mechanical significance of this property is very important. If an external load exists at point A, it can be included to the state vector by using this property.

5. The External Loads and Particular Solution

If an external load, exists in the domain as depicted in Fig. 5. For $z < \xi$ the state vector can be obtained as follows

$$\mathbf{S}(z) = \mathbf{F}(z) \cdot \mathbf{S}(0) \tag{29}$$

For $z > \xi$, by taking the point ξ^+ as the origin and $\mathbf{S}(\xi^-)$ at point ξ^- can be transferred to ξ^+ by writing the equilibrium equation on the beam segment. The value of the state vector at point ξ^+ will be

$$\mathbf{S}(\xi^+) = \mathbf{F}(\xi) \cdot \mathbf{S}(0) + \mathbf{K}(\xi)\,, \tag{30}$$

where $\mathbf{K}(\xi)$ is load vector and is defined as follows:

$$\mathbf{K}(\xi) = \begin{bmatrix} 0 \\ 0 \\ -M(\xi) \\ -P(\xi) \end{bmatrix} \tag{31}$$

The state vector at an arbitrary point $z > \xi$ can be obtained as

$$\mathbf{S}(z) = \mathbf{F}(z-\xi) \cdot \mathbf{S}(\xi^+) \tag{32}$$

And using the Eq. (30) and third property one obtains

$$\mathbf{S}(z) = \mathbf{F}(z)\mathbf{S}(0) + \mathbf{F}(z-\xi)\mathbf{K}(\xi),\ For\ \ z| > \xi (33)$$

If there is an external load at several points then

$$\mathbf{S}(z) = \mathbf{F}(z) \cdot \mathbf{S}(0) + \sum \mathbf{F}(z-\xi_i)\mathbf{K}(\xi_i) \tag{34}$$

If the external loads are distributed along the axes of the beam, then the summation is replaced by the following integral.

$$\mathbf{S}(z) = \mathbf{F}(z) \cdot \mathbf{S}(0) + \int_0^z \mathbf{F}(z-\xi)\mathbf{K}(\xi)d\xi \tag{35}$$

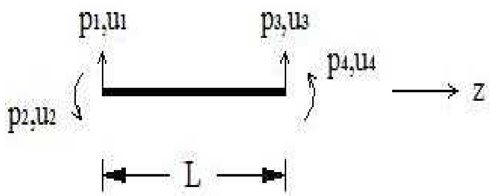

Figure 6. The positive direction of nodal forces and nodal displacements

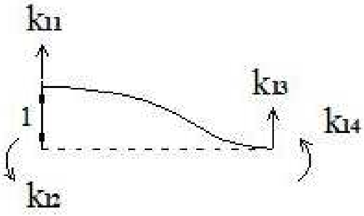

Figure 7. $u_1 = 1$, $u_2 = u_3 = u_4 = 0$ and the positive direction of k_{1i}

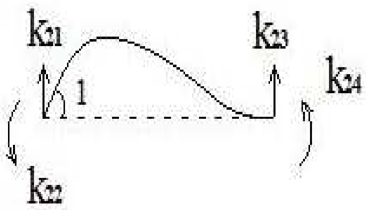

Figure 8. $u_2 = 1$, $u_1 = u_3 = u_4 = 0$ and the positive direction of k_{2i}

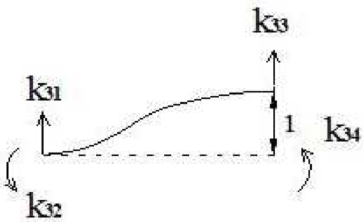

Figure 9. $u_3 = 1$, $u_1 = u_2 = u_4 = 0$ and the positive direction of k_{3i}

6. The derivation of stiffness matrix

The loads are acting the plane on the ends of the beam element, lying between two nods shown in Fig. 6. The aim is to express the nodal forces in terms of the nodal displacements. When nodal forces expressed in terms of nodal displacements, we have set of stiffness equations. The matrix form of stiffness equations as follows:

$$\mathbf{p} = \mathbf{K}.\mathbf{u}\,, \tag{36}$$

where $\mathbf{p}$, $\mathbf{u}$ are the nodal force and nodal displacement vectors respectively. $\mathbf{K}$ is the 4-rowed square matrix. The element of the stiffness matrix K_{ij} is the force at the point j, for the unit displacement at the point i. The total 16 stiffness coefficients correspond to the following the four deformation configurations of the beam.

Type I	$u_1 = 1,\ u_2 = u_3 = u_4 = 0$
Type II	$u_2 = 1,\ u_1 = u_3 = u_4 = 0$
Type III	$u_3 = 1,\ u_1 = u_2 = u_4 = 0$
Type IV	$u_4 = 1,\ u_1 = u_2 = u_3 = 0$

These four deformation configurations and the corresponding stiffness coefficients depicted in the Fig. 7 through the Fig. 10.

The unknown stiffness coefficients will be determined by using transfer matrix. The necessary transfer matrix equations for these cases are written in the Eq. (37),. through the Eq. (40).

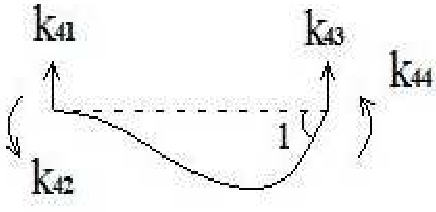

Figure 10. $u_4 = 1$, $u_1 = u_2 = u_3 = 0$ and the positive direction of k_{3i}

$$\begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} \\ F_{21} & F_{22} & F_{23} & F_{24} \\ F_{31} & F_{32} & F_{33} & F_{34} \\ F_{41} & F_{42} & F_{43} & F_{44} \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ -k_{12} \\ k_{11} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ k_{14} \\ -k_{13} \end{bmatrix} \quad (37)$$

$$\begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} \\ F_{21} & F_{22} & F_{23} & F_{24} \\ F_{31} & F_{32} & F_{33} & F_{34} \\ F_{41} & F_{42} & F_{43} & F_{44} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ -k_{22} \\ k_{21} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ k_{24} \\ -k_{23} \end{bmatrix} \quad (38)$$

$$\begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} \\ F_{21} & F_{22} & F_{23} & F_{24} \\ F_{31} & F_{32} & F_{33} & F_{34} \\ F_{41} & F_{42} & F_{43} & F_{44} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -k_{32} \\ k_{31} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ k_{34} \\ -k_{33} \end{bmatrix} \quad (39)$$

$$\begin{bmatrix} F_{11} & F_{12} & F_{13} & F_{14} \\ F_{21} & F_{22} & F_{23} & F_{24} \\ F_{31} & F_{32} & F_{33} & F_{34} \\ F_{41} & F_{42} & F_{43} & F_{44} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -k_{42} \\ k_{41} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ k_{44} \\ -k_{43} \end{bmatrix} \quad (40)$$

The unknown stiffness coefficients will be determined by solving these four matrix equations. The complete solution is given in the Eq. (41)

$$\mathbf{K} = \begin{bmatrix} K_{11} & K_{12} & -\frac{F_{23}}{\zeta} & -\frac{F_{13}}{\zeta} \\ K_{21} & K_{22} & -\frac{F_{24}}{\zeta} & -\frac{F_{14}}{\zeta} \\ -\frac{F_{23}}{\zeta} & -\frac{F_{24}}{\zeta} & K_{33} & K_{34} \\ -\frac{F_{13}}{\zeta} & -\frac{F_{14}}{\zeta} & K_{43} & K_{44} \end{bmatrix} \quad (41)$$

where
$K_{11} = \frac{F_{11}F_{23}-F_{13}F_{21}}{\zeta}$,
$K_{12} = \frac{F_{11}F_{24}-F_{14}F_{21}}{\zeta}$,
$K_{21} = \frac{F_{11}F_{24}-F_{14}F_{21}}{\zeta}$,
$K_{22} = \frac{F_{14}F_{22}-F_{12}F_{24}}{\zeta}$,
$K_{33} = \frac{F_{11}F_{23}-F_{13}F_{21}}{\zeta}$,
$K_{34} = \frac{F_{14}F_{21}-F_{11}F_{24}}{\zeta}$,
$K_{43} = \frac{F_{14}F_{21}-F_{11}F_{24}}{\zeta}$,
$K_{44} = \frac{F_{14}F_{22}-F_{12}F_{24}}{\zeta}$,
and here $\zeta$is

$$\zeta = F_{14}F_{23} - F_{13}F_{24} \quad (42)$$

The stiffness equation given in Eq. (41) is valid for three cases. But this equation will have the following simple form for $\alpha = 1$

$$k_{11} = \frac{k}{\lambda}\left(\frac{4z\lambda - 6\sinh 2z\lambda}{\eta}\right)$$

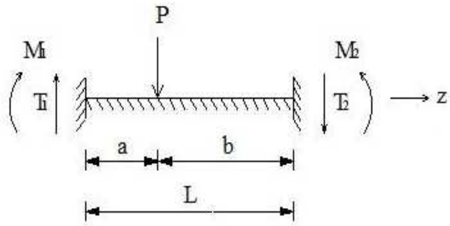

Figure 11. Single load and positive direction of internal forces

$$\begin{aligned}
k_{12} &= \frac{k}{\lambda^2}\left(\frac{\eta + 4z^2\lambda^2}{3\eta}\right) \\
k_{13} &= -\frac{4k}{\lambda}\left(\frac{z\lambda\cosh z\lambda - 3\sinh z\lambda}{\eta}\right) \\
k_{14} &= -\frac{4kz}{\lambda}\left(\frac{\sinh z\lambda}{\eta}\right) \\
k_{21} &= k_{12} \\
k_{22} &= -\frac{2k}{\lambda^3}\left(\frac{2z\lambda + 3\sinh 2z\lambda}{\eta}\right) \\
k_{23} &= -k_{14} \\
k_{23} &= \frac{4k}{\lambda^3}\left(\frac{z\lambda\cosh z\lambda + 3\sinh z\lambda}{\eta}\right) \\
k_{31} &= k_{13} \\
k_{32} &= k_{23} \\
k_{33} &= k_{11} \\
k_{34} &= -k_{12} \\
k_{41} &= k_{14} \\
k_{42} &= k_{24} \\
k_{43} &= k_{34} \\
k_{44} &= k_{22} \qquad (43)
\end{aligned}$$

where

$$\eta = 9 + 2z^2\lambda^2 - 9\cosh 2z\lambda \quad (44)$$

7. Fixed-end Loads

Since beams are frequently subject to forces acting between nods, it is necessary to derive fixed-end forces. For the common engineering applications, corresponding only two types of loads Fig. 11 and Fig. 12, the fixed-end forces will be obtained using transfer matrix.

Using Eq. (35) for $z = L$ and $\zeta = a$ we end up

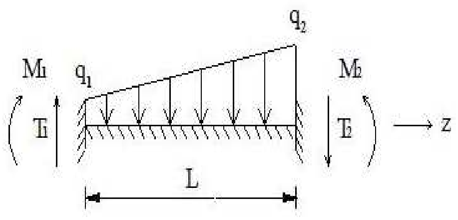

Figure 12. Distributed load and positive direction of internal forces

with

$$\begin{bmatrix} 0 \\ 0 \\ M_2 \\ T_2 \end{bmatrix} = F(L) \cdot \begin{bmatrix} 0 \\ 0 \\ M_1 \\ T_1 \end{bmatrix} + F(b) \cdot \begin{bmatrix} 0 \\ 0 \\ 0 \\ -P \end{bmatrix} \tag{45}$$

If we solve these equations for M_1, M_2, T_1, T_2 we will obtain

$$\begin{aligned} M_1 &= \frac{P\left\{F_{14}(L)F_{24}(b) - F_{24}(L)F_{14}(b)\right\}}{\left[F_{13}(L)\right]^2 + F_{14}(L)F_{23}(L)} \\ T_1 &= \frac{P\left\{F_{23}(L)F_{14}(b) + F_{13}(L)F_{13}(b)\right\}}{\left[F_{13}(L)\right]^2 + F_{14}(L)F_{23}(L)} \\ M_2 &= F_{33}(L)M_1 + F_{34}(L)T_1 - F_{34}(b)P \\ T_2 &= F_{43}(L)M_1 + F_{44}(L)T_1 - F_{44}(b)P \end{aligned} \tag{46}$$

Using Eq. (35) we have,

$$\begin{aligned} \mathbf{S}(z) &= \mathbf{F}(z) \cdot \mathbf{S}(0) \\ &- \int_0^L \mathbf{F}(z-\xi) \begin{bmatrix} 0 \\ 0 \\ 0 \\ (q_1 + m\xi) \end{bmatrix} d\xi \,, \end{aligned} \tag{47}$$

where

$$m = \frac{q_2 - q_1}{L} \tag{48}$$

If we insert boundary conditions into Eq. (47)

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \\ M_2 \\ T_2 \end{bmatrix} &= \mathbf{F}(L) \cdot \begin{bmatrix} 0 \\ 0 \\ M_1 \\ T_1 \end{bmatrix} \\ &- \int_0^L \begin{bmatrix} F_{14}(L-\xi) \\ F_{24}(L-\xi) \\ F_{34}(L-\xi) \\ F_{44}(L-\xi) \end{bmatrix} (q_1 + m\xi)\, d\xi \end{aligned} \tag{49}$$

Solving these equations for M_1, M_2, T_1, T_2 we have,

$$\begin{aligned} M_1 &= \frac{\{\overline{A}\}\, F_{13}(L) + \{A_5\}\, F_{14}(L)}{\left[F_{13}(L)\right]^2 + F_{14}(L)F_{23}(L)} \\ T_1 &= \frac{\{\overline{A}\}\, F_{23}(L) - \{A_5\}\, F_{13}(L)}{\left[F_{13}(L)\right]^2 + F_{14}(L)F_{23}(L)} \\ M_2 &= F_{33}(L)M_1 + F_{34}(L)T_1 + A_6 \\ T_2 &= F_{43}(L)M_1 + F_{44}(L)T_1 - A_7 \,, \end{aligned} \tag{50}$$

where $\overline{A} = A_1 + A_2 + A_3 + A_4$ and

$$\begin{aligned} A_1 &= \frac{1}{2EL}\left\{\Psi_8\,(1 - 2\cos 2\omega)\csc\omega\right\} B_1 \\ A_2 &= \frac{1}{2EL}\left\{\Psi_8\,(1 + 2\cos 2\omega)\sec\omega\right\} B_2 \\ A_3 &= \frac{1}{2EL}\left\{\Psi_9\,(\csc\omega)\right\} B_1 \\ A_4 &= -\frac{1}{2EL}\left\{\Psi_9\,(\sec\omega)\right\} B_2 \\ A_5 &= \frac{1}{2EL^2}\left(\frac{\Psi_7}{\sin 2\omega}\right) B_3 \\ A_6 &= -L \cdot \Psi_2\,(\Psi_4 \cdot \csc\omega \cdot B_1 \\ &+ \Psi_6 \cdot \sec\omega \cdot B_2) \\ A_7 &= \left(\frac{\Psi_1 - \cos 2\omega}{\sin 2\omega}\right) B_3 + B_4 \,, \end{aligned} \tag{51}$$

where all Ψ_i defined in Eq. (18).

$$\begin{aligned} g_1 &= \left(\frac{kR^*}{D^*}\right)^{1/4} \sin\omega \\ g_2 &= \left(\frac{kR^*}{D^*}\right)^{1/4} \cos\omega \\ B_1 &= \frac{1}{(g_1^2 + g_2^2)^2} \times \\ &\times \{m\,(-2\cos[Lg_1]\sinh[Lg_2]g_1g_2 \\ &+ \cosh[Lg_2]\sin[Lg_1](-g_1^2 + g_2^2) \\ &+ Lg_1(g_1^2 + g_2^2)) + q_1(g_1^2 + g_2^2)\times \\ &\times ((1 - \cos[Lg_1]\cosh[Lg_2])g_1 \\ &+ \sin[Lg_1]\sinh[Lg_2]g_2)\} \\ B_2 &= \frac{1}{(g_1^2 + g_2^2)^2} \times \\ &\times \{m\,(2\cosh[Lg_2]\sin[Lg_1]g_1g_2 \\ &- \cos[Lg_1]\sinh[Lg_2](g_1^2 - g_2^2) \\ &- Lg_2(g_1^2 + g_2^2)) + q_1(g_1^2 + g_2^2)\times \\ &\times (\sin[Lg_1]\sinh[Lg_2]g_1 \\ &+ (-1 + \cos[Lg_1]\cosh[Lg_2])g_2)\} \\ B_3 &= \frac{1}{(g_1^2 + g_2^2)^2} \times \\ &\times \{m\,(2g_1g_2 - 2\cos[Lg_1]\cosh[Lg_2]g_1g_2 \\ &- \sin[Lg_1]\sinh[Lg_2](g_1^2 - g_2^2)) \\ &+ q_1(g_1^2 + g_2^2)\,(-\cos[Lg_1]\sinh[Lg_2])g_1 \\ &+ \cosh[Lg_2]\sin[Lg_1]g_2)\} \end{aligned}$$

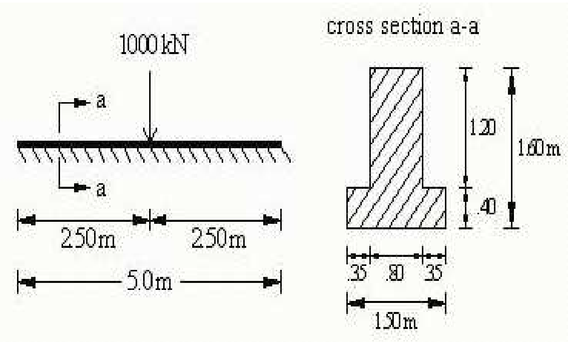

Figure 13. The beam and its cross-section

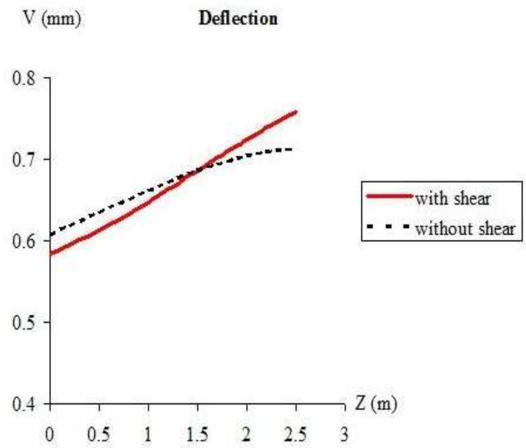

Figure 14. The vertical deflection of the beam

$$
\begin{aligned}
B_4 &= \frac{1}{(g_1^2+g_2^2)^2} \times \\
&\times \{m\,(2\sin[Lg_1]\sinh[Lg_2]g_1g_2 \\
&+ (g_1^2-g_2^2)(1-\cos[Lg_1]\cosh[Lg_2])) \\
&+ q_1(g_1^2+g_2^2)\,(\cosh[Lg_2]\sin[Lg_1])g_1 \\
&+ \cos[Lg_1]\sinh[Lg_2]g_2)\} \qquad (52)
\end{aligned}
$$

8. Applications

To evaluate the performance of the theory, the following three structures will be analyzed and the results will be compared with the existing solutions in the literature for Timoshenko beam and Bernoulli-Navier beam.

8.1. The Beam Subjected a Single Load

The behavior of free-ended beam of T-cross section and length l, carrying a single load at the middle will now considered Fig. 13.

Physical and geometric properties of the beam as follows:

E = 25 GPa	(Elastic modulus)
k = 300 MPa	(Subgrade modulus)
L = 5 m	(Span length)
P = 1000 kN	(Single force)
s = 2	(Shape factor of T-beam)
v = 0.2	(Poisson's ratio of the beam)
A = 1.56 m^2	(Area of the beam)
I = 0.36 m^4	(Moment of inertia)

This problem is solved with and without shear effect using transfer matrix. The transfer matrix without shear effect is obtained by Çengel [24]. The transfer matrix for the beam with shear effect the Eq. (17) is used. Because of single force acted at the middle of the beam, the following singular matrix is employed.

$$
\mathbf{K}(\xi) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -P \end{bmatrix} \qquad (53)
$$

The state vector **S** obtained as follow [16]

$$
\mathbf{S}(z) = \mathbf{F}(z)\cdot\mathbf{S}(0) + \mathbf{F}(z-\xi)\cdot\mathbf{K}(\xi) \qquad (54)
$$

The deflection curves with and without shear effect are obtained and they are depicted in Fig. 14.

The shear does not effect the internal moment and shear force distribution. Therefore, only the moment and the shear force distributions given in Fig. 15 and Fig. 16 respectively.

8.2. The Beams Supporting Two Single Loads at the Ends

In this application, the behavior of a free-ended beam of T-cross section and length l, carrying two single loads at the ends, will considered for the following three different cases Fig. 17.

1. The first case: h = 1.2 m, k = 300 MPa and A = 1.24 m^2, I = 0.154 m^4
2. The second case: h = 1.6 m, k = 300 MPa and A = 1.56 m^2, I = 0.360 m^4

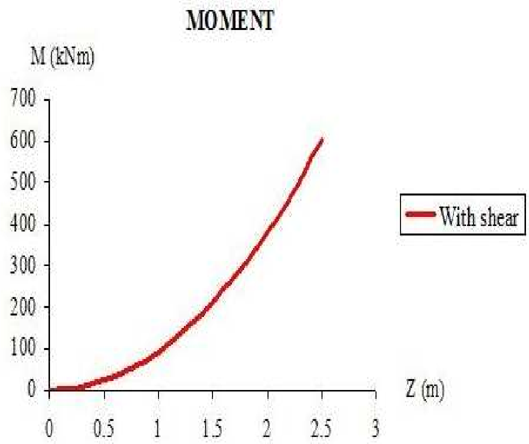

Figure 15. The moment distribution

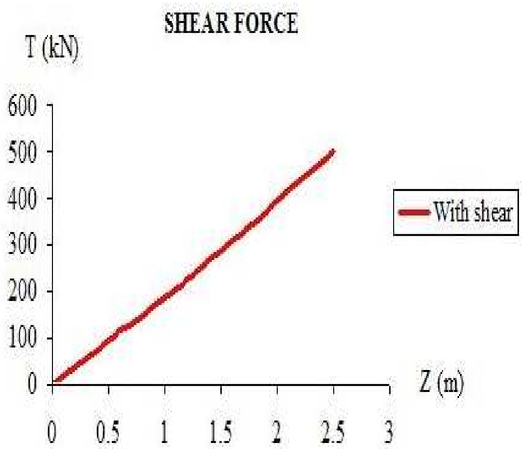

Figure 16. The shear force distribution

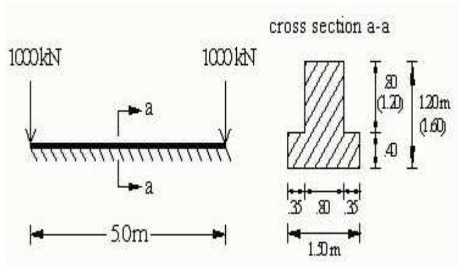

Figure 17. The beam and its cross-section

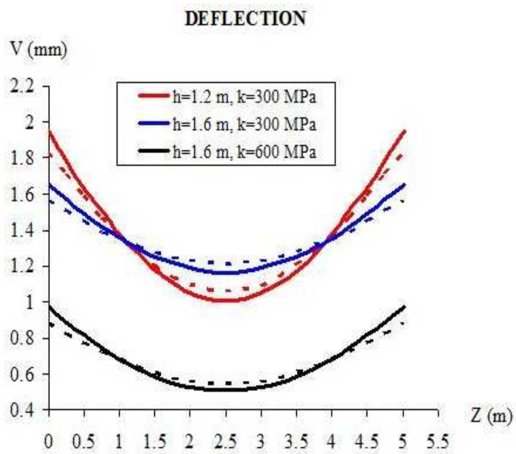

Figure 18. The deflection curves for three different cases with and without shear

3. The third case: h = 1.6 m, k = 600 MPa and A = 1.56 m^2, I = 0.360 m^4

In the all three cases the modulus of elasticity is E=25 GPa, Poisson's ratio is v=0.2, shape factor is s=2.

For this problem α is grater than 1, therefore the transfer matrix will be used given by Eq. (17). For the case, without shear effect the transfer matrix is employed which is given by Çengel [24]. The same problems are solved by Aydoğan [13] using FEM analysis. Aydoğan gives the values at nodal points. In this work the values of deflection and internal forces are obtained as continuous functions. Deflections curves are given in Fig. 18. Dotted curves show the deflection without shear effect.

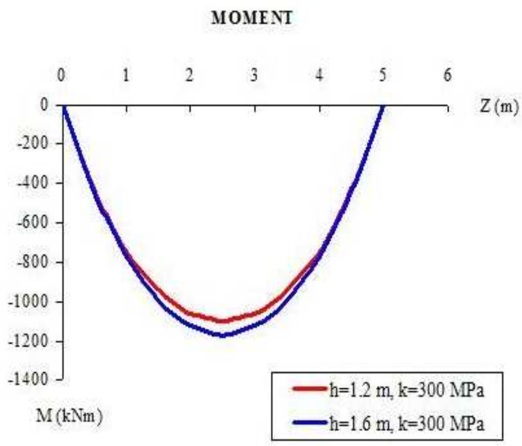

Figure 19. Moment curves for two types of beams

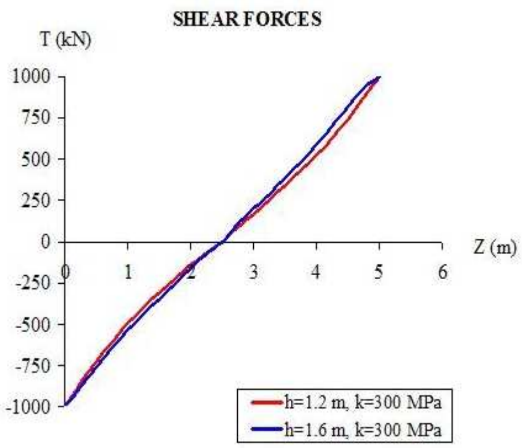

Figure 20. The shear force curves for two type beams

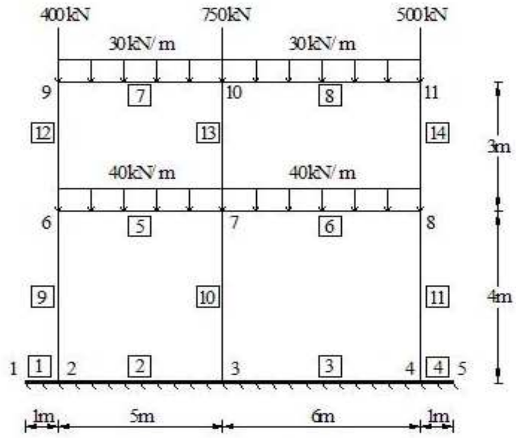

Figure 21. The structure geometry and its loading

Internal forces; moment and shear force diagrams are shown in Fig. 19 and Fig. 20, respectively.

8.3. The Solution of Two-Story Building

The last structure considered is two-story frame whose dimensions and loading shown in Fig. 21. The shape factor, s is taken as 2. The material properties of the frame are E=25 GPa. v=0.2. This building supported to continuous beam on elastic foundation with sub grade modulus 450 MPa.

The structure geometry, loads, nodal and element numbers are depicted in Fig. 21. The beam's cross-sections are given in Fig. 22. This problem is solved for both; with shear effects and without shear effects. In the solution with shear effect, the stiffness matrix is used given in Eq. (41). For the solution without shear effect, the stiffness matrix is obtained by Çengel [24].

The deflection curves of foundation are shown in Fig. 23. The difference in between the deflection curves with shear effect and without shear effect is very clear. Especially discontinuity of first derivative of displacement curve under singular force is very interesting result, which can be directly concluded from Eq. (1). The internal forces diagrams are depicted Fig. 24 and Fig. 25 for only with shear effect. Because of the effect of shear on the distribution of internal forces is negligible.

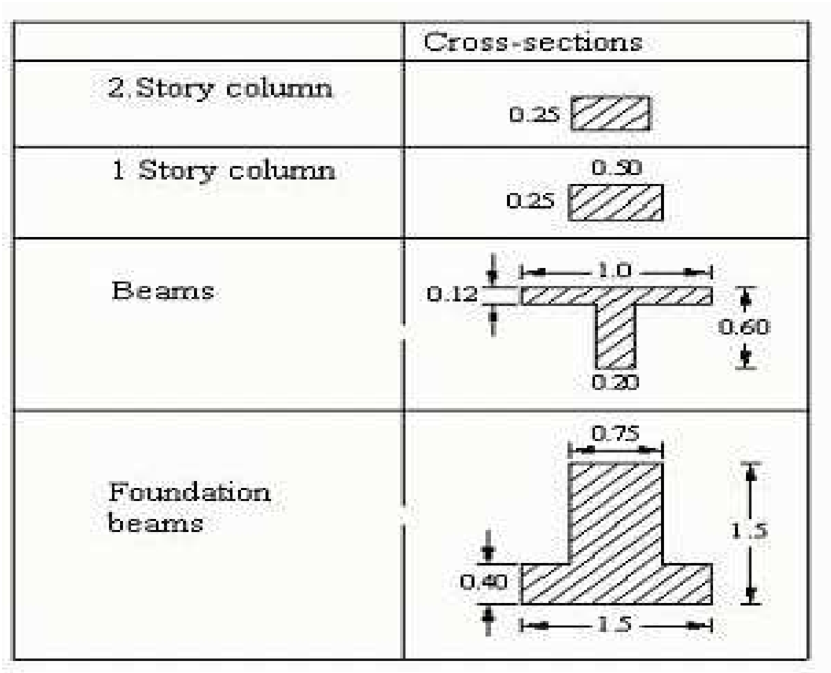

Figure 22. The beam's cross-sections

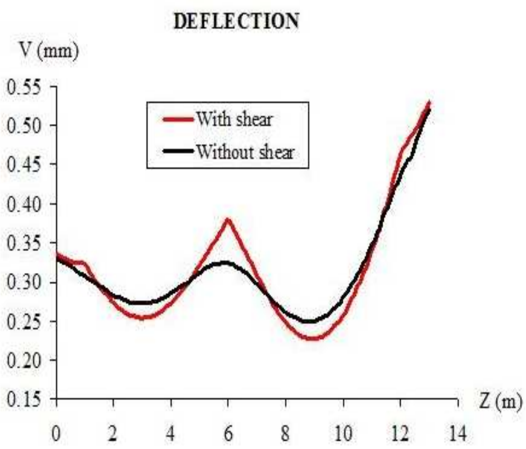

Figure 23. The deflection curves of the foundation

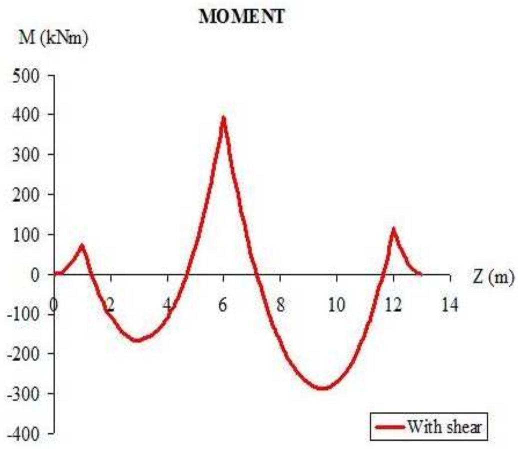

Figure 24. The moment diagram of the foundation

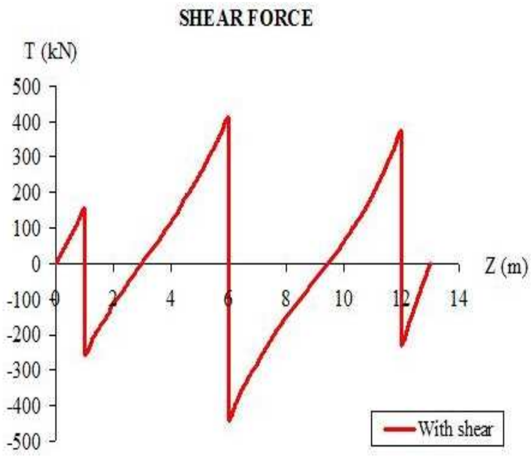

Figure 25. The shear force diagram of the foundation

9. Conclusion

In this study, the effect of shear is investigated for the beams on elastic foundation. For the solution three different transfer matrix are derived depending on parameter, α.

1. Three different stiffness matrixes are derived depending on parameter α.
2. Fixed-end loads are derived for two load conditions, which may be helpful for engineering applications.
3. The effect of shear on deflection is meaningful.
4. The effect of shear increases with increasing height of the beam.
5. The effect of shear on the distribution of internal forces is negligible for engineering purpose.
6. The deflection decreases with the increasing sub grade modulus and height of beam as expected. In this variation, sub grade modulus is more effective than the height of the beam.

References

[1] S.P. Timoshenko and D.Y. Young, Vibration problems in engineering (New York, 1961).

[2] A.D. Kerr, J. Appl. Mech. Trans. ASME **31**, 491 (1964).

[3] M. Hetenyi, Beams on elastic foundation (University of Michigan Press Ann Arbor, MI, 1946).

[4] A. Dodge, J. Struct. Div. ASCE **90**, 63 (1964).

[5] C.K. Miranda and K. Nair, J Struct. Div. ASCE **92**, 131 (1966).

[6] F. Keskinel Elastik zemine outran dikdörtgen düzlem kapalı çerçeveler, ITU Mimarlık Fakültesi (1967).

[7] K.J. Bathe, Finite element procedure in engineering analysis (Prentice-Hall, Englewood Cliffs, NJ. 1982).

[8] J.N. Reddy, Finite element method (McGraw-Hill, Inc 1993).

[9] A.Y. Aköz, M.H. Omurtag, and A.N. Doğruoğlu, Int. J. Solids Structures **28**, 225 (1990).

[10] Z. Friedman and J.B. Kosmata, Int. J. for Numerical Methods in Engineering **41**, 473 (1998).

[11] J. Jou and S. Yang, Applied Mathematics and Computations **115**, 63 (2000).

[12] A.Y. Aköz and F. Kadıoğlu, Computer and Structures **60**, 643 (1996).

[13] M. Aydoğan, Journal of Structural Engineering, 1270, (1995).

[14] M. İnan, The Bulletin of the Technical University of Istanbul **14** (1961).

[15] V. Cinemre, The Bulletin of the Technical University of Istanbul **14** (1961).

[16] M. İnan, The Bulletin of the Technical University of Istanbul **14** (1961).

[17] M. İnan, Technical University of Istanbul (1964).

[18] S. Zubaroğlu, MSc. Thesis, Istanbul Technical University (1994).

[19] J-H Yin, J.Geotech and Envir Engr. ASCE **126**, 265 (2000).

[20] J-H Yin, Journal of Engineering Mechanics **126** (2000).

[21] H. Antes, Computer and Structures **81**, 383 (2003).

[22] S. A Alghamdi, Computer and Structures **79**, 1175 (2001).

[23] V. Cinemre, Linear Algebra, Technical University of Istanbul, (1964)

[24] B. Çengel, MSc. Thesis, Istanbul Technical University (2003).

A. The Idempotents for the Case $\alpha > 1$

The following properties of idempotents are helpful for the mathematical manipulations:

1. $\sum_{i=1}^{N} \mathbf{P}_i = \mathbf{I}$
2. $\mathbf{P}_i \mathbf{P}_i = \mathbf{P}_i$

3. $\mathbf{P}_i\mathbf{P}_j = 0 \quad for \quad i \neq j$

We need the four matrix equations to solve the four idempotent. The four matrix equations as follows [22]:

$$\begin{aligned}
&\mathbf{P}_1 + \mathbf{P}_2 + \mathbf{P}_3 + \mathbf{P}_4 = \mathbf{I} \\
&\lambda_1\mathbf{P}_1 + \lambda_2\mathbf{P}_2 + \lambda_3\mathbf{P}_3 + \lambda_4\mathbf{P}_4 = \mathbf{D} \\
&\lambda_1^2\mathbf{P}_1 + \lambda_2^2\mathbf{P}_2 + \lambda_3^2\mathbf{P}_3 + \lambda_4^2\mathbf{P}_4 = \mathbf{D}^2 \\
&\lambda_1^3\mathbf{P}_1 + \lambda_2^3\mathbf{P}_2 + \lambda_3^3\mathbf{P}_3 + \lambda_4^3\mathbf{P}_4 = \mathbf{D}^3 \,,
\end{aligned} \tag{55}$$

where $\mathbf{D}$ is differential transmitter matrix, $\mathbf{P}_i$ are idempotent matrix and $\mathbf{I}$ is unit matrix.

The mathematical manipulations are omitted for sake of simplicity. The reader may refer the reference [23]

$\mathbf{P}_1$ The elements of first idempotent:

$$\begin{aligned}
P_{11} &= B\left(\mathrm{e}^{-2i\omega} - \mu\sqrt{\frac{k}{D^*R^*}}\right) \\
P_{12} &= \frac{B\mathrm{e}^{-i\omega}\left(\mu\sqrt{k} - \mathrm{e}^{-2i\omega}\sqrt{D^*R^*}\right)}{(kD^*)^{1/4}(R^*)^{3/4}} \\
P_{13} &= B\sqrt{\frac{R^*}{kD^*}} \\
P_{14} &= B\mathrm{e}^{-i\omega} \times \\
&\times \frac{\left\{D^*R^* + \mu\sqrt{k}\left(\mathrm{e}^{-2i\omega}\sqrt{D^*R^*} - \mu\sqrt{k}\right)\right\}}{(D^*)^{5/4}(kR^*)^{3/4}} \\
P_{21} &= -B\mathrm{e}^{-i\omega}\left(\frac{kR^*}{D^*}\right)^{1/4} \\
P_{22} &= B\mathrm{e}^{-2i\omega} \\
P_{23} &= \frac{B\mathrm{e}^{-3i\omega}}{k^{1/4}}\left(\frac{R^*}{D^*}\right)^{3/4} \\
P_{24} &= -B\sqrt{\frac{R^*}{kD^*}} \\
P_{31} &= -B\sqrt{\frac{kD^*}{R^*}} \\
P_{32} &= B\mathrm{e}^{-i\omega}(k^{1/4})\left(\frac{D^*}{R^*}\right)^{3/4} \\
P_{33} &= B\mathrm{e}^{-2i\omega} \\
P_{34} &= \frac{B\mathrm{e}^{-i\omega}\left(\mathrm{e}^{-2i\omega}\sqrt{D^*R^*} - \mu\sqrt{k}\right)}{(kD^*)^{1/4}(R^*)^{3/4}} \\
P_{41} &= B\mathrm{e}^{-i\omega}(k^{3/4}) \times \\
&\times \frac{\left(\mathrm{e}^{-2i\omega}\sqrt{D^*R^*} - \mu\sqrt{k}\right)}{(D^*)^{1/4}(R^*)^{3/4}} \\
P_{42} &= B\sqrt{\frac{kD^*}{R^*}} \\
P_{43} &= B\mathrm{e}^{-i\omega}\left(\frac{kR^*}{D^*}\right)^{1/4} ,
\end{aligned} \tag{56}$$

where γ is Poisson ratio, r is the radius of moment of inertia

$$\begin{aligned}
B &= \frac{i}{4 \cdot \sin 2\omega} \\
\mu &= 2s(1 + v) \\
D^* &= EA \\
R^* &= \frac{1}{r^2} = \frac{A}{I} \\
P_{44} &= B\left(\mathrm{e}^{-2i\omega} - \mu\sqrt{\frac{k}{D^*R^*}}\right)
\end{aligned} \tag{57}$$

$\mathbf{P}_2$ The elements of second idempotent:

$$\begin{aligned}
P_{11} &= B\left(\mathrm{e}^{-2i\omega} - \mu\sqrt{\frac{k}{D^*R^*}}\right) \\
P_{12} &= -\frac{B\mathrm{e}^{-i\omega}\left(\mu\sqrt{k} - \mathrm{e}^{-2i\omega}\sqrt{D^*R^*}\right)}{(kD^*)^{1/4}(R^*)^{3/4}} \\
P_{13} &= B\sqrt{\frac{R^*}{kD^*}} \\
P_{14} &= -B\mathrm{e}^{-i\omega} \times \\
&\times \frac{\left\{D^*R^* + \mu\sqrt{k}\left(\mathrm{e}^{-2i\omega}\sqrt{D^*R^*} - \mu\sqrt{k}\right)\right\}}{(D^*)^{5/4}(kR^*)^{3/4}} \\
P_{21} &= B\mathrm{e}^{-i\omega}\left(\frac{kR^*}{D^*}\right)^{1/4} \\
P_{22} &= B\mathrm{e}^{-2i\omega} \\
P_{23} &= -\frac{B\mathrm{e}^{-3i\omega}}{k^{1/4}}\left(\frac{R^*}{D^*}\right)^{3/4} \\
P_{24} &= -B\sqrt{\frac{R^*}{kD^*}} \\
P_{31} &= -B\sqrt{\frac{kD^*}{R^*}} \\
P_{32} &= -B\mathrm{e}^{-i\omega}(k^{1/4})\left(\frac{D^*}{R^*}\right)^{3/4} \\
P_{33} &= B\mathrm{e}^{-2i\omega} \\
P_{34} &= -\frac{B\mathrm{e}^{-i\omega}\left(\mathrm{e}^{-2i\omega}\sqrt{D^*R^*} - \mu\sqrt{k}\right)}{(kD^*)^{1/4}(R^*)^{3/4}} \\
P_{41} &= -B\mathrm{e}^{-i\omega}(k^{3/4}) \times
\end{aligned}$$

$$\times \quad \frac{\left(e^{-2i\omega}\sqrt{D^*R^*} - \mu\sqrt{k}\right)}{(D^*)^{1/4}(R^*)^{3/4}}$$

$$P_{42} = B\sqrt{\frac{kD^*}{R^*}}$$

$$P_{43} = -Be^{-i\omega}\left(\frac{kR^*}{D^*}\right)^{1/4}$$

$$P_{44} = B\left(e^{-2i\omega} - \mu\sqrt{\frac{k}{D^*R^*}}\right) \tag{58}$$

$\mathbf{P}_3$ The elements of third idempotent:

$$P_{11} = -B\left(e^{2i\omega} - \mu\sqrt{\frac{k}{D^*R^*}}\right)$$

$$P_{12} = -\frac{Be^{i\omega}\left(\mu\sqrt{k} - e^{2i\omega}\sqrt{D^*R^*}\right)}{(kD^*)^{1/4}(R^*)^{3/4}}$$

$$P_{13} = -B\sqrt{\frac{R^*}{kD^*}}$$

$$P_{14} = -Be^{i\omega} \times$$

$$\times \frac{\left\{D^*R^* + \mu\sqrt{k}\left(e^{2i\omega}\sqrt{D^*R^*} - \mu\sqrt{k}\right)\right\}}{(D^*)^{5/4}(kR^*)^{3/4}}$$

$$P_{21} = Be^{i\omega}\left(\frac{kR^*}{D^*}\right)^{1/4}$$

$$P_{22} = -Be^{2i\omega}$$

$$P_{23} = -\frac{Be^{3i\omega}}{k^{1/4}}\left(\frac{R^*}{D^*}\right)^{3/4}$$

$$P_{24} = B\sqrt{\frac{R^*}{kD^*}}$$

$$P_{31} = B\sqrt{\frac{kD^*}{R^*}}$$

$$P_{32} = -Be^{i\omega}(k^{1/4})\left(\frac{D^*}{R^*}\right)^{3/4}$$

$$P_{33} = -Be^{2i\omega}$$

$$P_{34} = \frac{Be^{i\omega}\left(\mu\sqrt{k} - e^{2i\omega}\sqrt{D^*R^*}\right)}{(kD^*)^{1/4}(R^*)^{3/4}}$$

$$P_{41} = Be^{i\omega}(k^{3/4}) \times$$

$$\times \quad \frac{\left(\mu\sqrt{k} - e^{2i\omega}\sqrt{D^*R^*}\right)}{(D^*)^{1/4}(R^*)^{3/4}}$$

$$P_{42} = -B\sqrt{\frac{kD^*}{R^*}}$$

$$P_{43} = -Be^{i\omega}\left(\frac{kR^*}{D^*}\right)^{1/4}$$

$$P_{44} = -B\left(e^{2i\omega} - \mu\sqrt{\frac{k}{D^*R^*}}\right) \tag{59}$$

$\mathbf{P}_4$ The elements of fortht idempotent:

$$P_{11} = -B\left(e^{2i\omega} - \mu\sqrt{\frac{k}{D^*R^*}}\right)$$

$$P_{12} = \frac{Be^{i\omega}\left(\mu\sqrt{k} - e^{2i\omega}\sqrt{D^*R^*}\right)}{(kD^*)^{1/4}(R^*)^{3/4}}$$

$$P_{13} = -B\sqrt{\frac{R^*}{kD^*}}$$

$$P_{14} = Be^{i\omega} \times$$

$$\times \frac{\left\{D^*R^* + \mu\sqrt{k}\left(e^{2i\omega}\sqrt{D^*R^*} - \mu\sqrt{k}\right)\right\}}{(D^*)^{5/4}(kR^*)^{3/4}}$$

$$P_{21} = -Be^{i\omega}\left(\frac{kR^*}{D^*}\right)^{1/4}$$

$$P_{22} = -Be^{2i\omega}$$

$$P_{23} = \frac{Be^{3i\omega}}{k^{1/4}}\left(\frac{R^*}{D^*}\right)^{3/4}$$

$$P_{24} = B\sqrt{\frac{R^*}{kD^*}}$$

$$P_{31} = B\sqrt{\frac{kD^*}{R^*}}$$

$$P_{32} = Be^{i\omega}(k^{1/4})\left(\frac{D^*}{R^*}\right)^{3/4}$$

$$P_{33} = -Be^{2i\omega}$$

$$P_{34} = -\frac{Be^{i\omega}\left(\mu\sqrt{k} - e^{2i\omega}\sqrt{D^*R^*}\right)}{(kD^*)^{1/4}(R^*)^{3/4}}$$

$$P_{41} = -Be^{i\omega}(k^{3/4}) \times$$

$$\times \quad \frac{\left(\mu\sqrt{k} - e^{2i\omega}\sqrt{D^*R^*}\right)}{(D^*)^{1/4}(R^*)^{3/4}}$$

$$P_{42} = -B\sqrt{\frac{kD^*}{R^*}}$$

$$P_{43} = Be^{i\omega}\left(\frac{kR^*}{D^*}\right)^{1/4}$$

$$P_{44} = -B\left(e^{2i\omega} - \mu\sqrt{\frac{k}{D^*R^*}}\right) \tag{60}$$