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Critical Problem for an Infinite Cylinder with Forward Scattering

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The transport of monoenergetic neutrons in a bare homogeneous cylinder is studied. The transport equation is solved using F_N method considering the pseudo-problem. Numerical values for the critical radius are obtained and tabulated for various c values. Numerical results indicate that the critical radius varies non-monotonically with forward scattering. Some selected illustrative results are compared with those already available in the literature. It is also shown that the F_N method, though approximate, yields results accurate to at least three or four significant figures for the problem considered.

Keywords: Neutron transport theory, forward and backward scattering, critical radius

1. Introduction

The transport of monoenergetic neutrons in spheres, infinite slab and cylinders has been the subject of many studies. The critical problems for isotropic scattering have been obtained by using various methods in different geometries [1-12]. Some of the work included anisotropic scattering [4-6,8]. The problem of anisotropy in the interactions of neutrons with matter and of its effect on the critical size is one of the most important problems of neutron transport theory. When anisotropy is taken into account, some new problems occur in transforming the transport equation into more conventional one. A large variety of numerical methods have been developed which appear quite different, and based on a few approximation techniques, such as a finite differences for differential operators, quadrature formulas for integral operators, or surface-integral equation methods as the complementary method [3] and the facile method [13-18].

In more recent works, Thomas et al. [18], and Siewert et al. [13] have reported F_N solutions to one-speed problems in cylindrical geometry for isotropic scattering. The F_N solution was found to be accurate to 5 or 6 significant figures for all cases. In the other works the integral transform method [2] has been extended to the treatment of one-dimensional homogeneous media with linearly anisotropic scattering. Their accuracy is very high, sometimes 12 significant digits are given. Also some numerical results for fundamental and higher eigenvalues have been obtained by Sjöstrand [9] using the formalizm of Sanchez and Ganapol [4].

In this paper, first we make the change of variables [19], then use the transformation idea developed by Mitsis [20], the elementary solutions of Case [21], and the F_N method [13-18,22] to solve the cylindrical critical problem. The two types of results will be presented. The first category shows the applicability of the proposed method as applied to the calculation of the criticality eigenvalues. The second concerns the investigation of the variation of the radius with forward scattering. For this purpose we fix the criticality factor c and α and determine the critical radius by simple iterative techniques. Furthermore, a comparison is also made with published results [2-6,11,12].

2. The Statement of the Problem

The starting stationary transport equation for neutrons of one speed and the angular flux $\Phi(\rho,\Omega)$ can be written [1,18,21]

$$\left[\vec{\Omega}\cdot\vec{\nabla}+\Sigma_t\right]\Phi(\rho,\Omega) = = c\Sigma_t\int_\Omega f(\Omega'\to\Omega)\Phi(\rho,\Omega')d\Omega' \tag{1a}$$

where

$$\vec{\Omega}\cdot\vec{\nabla}=\sin\theta\left(\cos\phi\frac{\partial}{\partial\rho}-\frac{sin\phi}{\rho}\frac{\partial}{\partial\phi}\right)\ . \tag{1b}$$

$f(\Omega' \to \Omega)$ is the scattering function, i.e, the probability that a neutron with direction Ω' before a collision has the direction Ω after the collision, and normalized to unity. For the proposed problem, the scattering function is given by [5,6,19,23-25]

$$f(\Omega' \to \Omega) = \frac{1-\alpha}{4\pi} + \alpha\delta(\Omega' - \Omega)\ . \tag{2}$$

Introducing the scattering function given by Eq. 2 into Eq. 1, the transport equation for a circular cylinder of radius ρ and infinite hight can be written as [1,3,8,18,20]

$$\sin\theta\left(\cos\phi\frac{\partial}{\partial r} - \frac{sin\phi}{r}\frac{\partial}{\partial\phi}\right)\Psi(r,\theta,\phi) + \Psi(r,\theta,\phi)$$
$$= \frac{c'}{4\pi}\int\Psi(r,\theta,\phi)\sin\theta d\theta d\phi \tag{3}$$

where the modified scattering parameters are [6,19]:

$$\rho\Sigma_t \to \rho \tag{4a}$$

$$with \quad (1-c\alpha)\rho = R \tag{4b}$$

$$c' = \frac{c(1-\alpha)}{1-\alpha c} \tag{4c}$$

and

$$\Phi(\rho,\Omega) = \Psi(r,\Omega)\ . \tag{5}$$

In writing Eq. 3, we have used standard neutronic notation, consistent with recent publications in the literature [19,24]. As already mentioned in the introduction the problem under investigation can be solved by considering the pseudo-problem [20,21,26]. Then, the integral equation for the neutron flux distribution Ψ can be written as [20],

$$\Psi(r) = c'\int_0^1\left[K_0(r/\mu)\int_0^r t\Phi(t)I_0(t/\mu)dt + \right.$$
$$\left. +I_0(r/\mu)\int_r^R t\Phi(t)K_0(t/\mu)dt\right]\frac{d\mu}{\mu^2} \tag{6}$$

where $I_0(r/\mu)$ and $K_0(r/\mu)$ denote the modified Bessel Functions. Proceeding to derive a Pseudo-problem we can solve to obtain $\Phi(r)$, following Mitsis [20]; we let

$$\Phi(r,\mu) = c'\left[K_0(r/\mu)\int_0^r t\Phi(t)I_0(t/\mu)dt + \right.$$
$$\left. +I_0(r/\mu)\int_r^R t\Phi(t)K_0(t/\mu)dt\right] \tag{7}$$

so that

$$\Phi(r) = \int_0^1 \Phi(r/\mu)\frac{d\mu}{\mu^2}\ . \tag{8}$$

Eq. 8 suggests that $\Phi(r,\mu)$ can be interpreted as a pseudo neutron distribution taking the place of $\Psi(r,\theta,\Phi)$. Differentiating Eq. 7, we find that $\Phi(r,\mu)$ for $\mu \in [0,1]$ and $r \in [0,R]$ satisfies

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{\mu}\right]\Phi(r,\mu) = -c'\int_0^1\Phi(r,\mu')\frac{d\mu'}{\mu'^2} \tag{9}$$

We can also deduce from Eq. 7 the boundary condition, for $\mu \in [0,1]$

$$K_1(R/\mu)\Phi(R,\mu) +$$
$$+\mu K_0(R/\mu)\frac{\partial}{\partial r}\Phi(r,\mu)\Big|_{r=R} = 0\,. \tag{10}$$

We can also express the solution of the Eq. 9 as

$$\Psi(r,\mu) = \mu^2\left\{A(\nu_0')\left[\Phi(\nu_0',\mu) + \Phi(-\nu_0',\mu)\right]\times\right.$$
$$\times I_0(r/\nu_0') + \int_0^1 A(\nu')\left[\Phi(\nu',\mu) + \Phi(-\nu',\mu)\right]\times$$
$$\left.\times\, I_0(r/\nu')d\nu\right\}\ , \tag{11}$$

where $A(\nu_0)$ and $A(\nu)$ are expansion coefficients to be determined from the boundary conditions and $\Phi(\pm\xi,\mu)$, $\xi \in P = \nu_0' \cup [0,1]$ with

$$\nu_0' = (1-\alpha c)\nu_0 \tag{12a}$$

Table 1
The critical radius ρ in mean-free-paths for isotropic scattering and comparison with published values

c	Present work	Ref.[4]	Ref.[2]	Ref.[12]	Ref.[3]	Ref.[11]	Ref.[24]
1.01	13.12552	13.1255155			13.12550		
1.02	9.043252		9.04458	9.0494		9.0433	
1.05	5.411283		5.41152	5.4140		5.4118	
1.10	3.577384	3.5773913	3.57744	3.5795	3.577383	3.5783	
1.2	2.287993		2.28724	2.2891	2.287201	2.2884	
1.3	1.725063	1.7250029			1.725006		
1.4	1.397116		1.39699	1.3991	1.396979	1.3973	
1.5	1.178365	1.178340			1.178342		
1.6	1.020883		1.02085	1.0231	1.020840	1.0209	
1.7	0.901406				0.901396		
1.8	0.807497		0.80743	0.8097	0.807428	1.8067	
1.9	0.731446				0.731430		
2	0.668618	0.6686129	0.66862	0.6709	0.668614	0.6673	
2.3	0.53183				0.531179		
2.5	0.46793				0.467924		
2.39318	0.50000						0.500
3.94239	0.25000						0.250
5.98205	0.15000						0.150
8.50732	0.10000						0.100
16.0109	0.05019						0.050
30.901	0.02446						0.025

$$\Phi(\pm\nu', \mu) = \frac{c'}{2}\nu' Pv\left(\frac{1}{\nu' \mp \mu}\right) + \left(1 - c'\nu' \tan h^{-1}\nu'\right)\partial(\nu' \mp \mu) \quad (12b)$$

and

$$\Phi(\pm\nu'_0, \mu) = \frac{c'}{2}\nu'_0\left(\frac{1}{\nu'_0 \mp \mu}\right) \quad (12c)$$

are the familiar (generalized) functions appropriate to one-speed neutron tranport theory in plane geometry problems. Here the discrete eigenvalues $\pm\nu'_0$ are the positive solutions of

$$c'\nu'_0 \tan h^{-1}(1/\nu'_0) - 1 = 0\,. \quad (13)$$

3. The F_N Solution

Having established the solution of Eq. 9, we note at this point that a discussion of the basic aspects of the F_N method has been reported in Ref. [13,18,19] where the approach is the same as used in previous works [13,18,19]. Since the fundamentals of the method and its development are so well-known, our presentation here is brief. Then following the procedure of Ref.[18,24] we derive a system of singular integral equations and constraints for unknown angular fluxes in more direct way, by using the completeness and full-range orthogonality properties of the generalized functions $\Phi(\xi, \mu)$:

$$\int_0^1 [\Phi(\xi,\mu) - \Phi(-\xi,\mu)]\,\{\mu + \xi B_1(R/\mu)B(R/\xi)\} \times \Psi(R,\mu)\frac{d\mu}{\mu} = 0\,, \quad (14)$$

where

$$B_1(R/\mu) = \frac{K_1(R/\mu)}{K_0(R/\mu)} \quad (15a)$$

and

$$B(R/\xi) = \frac{I_0(R/\xi)}{I_1(R/\xi)}\,. \quad (15b)$$

We now introduce the approximation

Table 2
The critical radius ρ in mean-free-paths for various degree of forward scattering and comparison with published values

α	c=1.1	α	c=1.2	α	c=2	Ref.[6] c=1.1	Ref.[5] c=1.1
0.1	3.738303	0.1	1.792517	0.05	0.67825		
0.2	3.924484	0.2	1.86909	0.1	0.68826		
0.3	4.143486	0.3	1.95666	0.15	0.69845		
0.4	4.406905	0.4	2.05844	0.2	0.70886		
0.5	4.730989	0.5	2.17766	0.3	0.7297		
0.6	5.146192	0.6	2.3167	0.4	0.7476		
0.7	5.70383	0.7	2.4643	0.45	0.7484		
0.7614	6.15677	0.75	2.482				6.1568
0.8438	6.9658	0.76	2.50			6.9612	6.9659
0.8846	7.4200					7.4168	7.4201
0.8970	7.5080					7.5045	7.5073
0.8994	7.5041					7.5099	7.5126
0.9018	7.5081					7.5078	7.5103
0.9042	7.4950					7.4946	7.4971
0.9079	7.4611					7.4316	7.4338

$$\Psi(R/\mu) = \mu^2 \sum_{k=0}^{N} a_k \mu^k \tag{16}$$

into Eq. 14 to obtain, for $\xi \in P$, a system of linear algebraic equations:

$$\sum_{k=0}^{N} a_k \left[E_k(\xi_\beta) + B(R/\xi_\beta) D(\xi_\beta)\right] = 0 \tag{17}$$

where

$$E_k(\xi) = \frac{1}{\xi}\int_0^1 \mu^{k+1}\left[\Phi(\xi,\mu) - \Phi(-\xi,\mu)\right] d\mu \tag{18a}$$

and

$$D_k(\xi) = \int_0^1 \mu^k \left[\Phi(\xi,\mu) - \Phi(-\xi,\mu)\right] B_1(R/\mu) d\mu \tag{18b}$$

are the functions discussed previously in Ref.[13,18]. Subsequently, we consider Eq. 17 at a set of collocation points $\xi = \xi_\beta$,

$$\xi_\beta = \frac{1}{2} + \frac{1}{2}\cos\left(\frac{2\beta - 1}{2N}\pi\right), \ \beta = 1, 2, ..., N\ . \tag{19}$$

Thus, to find the critical radius R [or $c = (\alpha c' - \alpha + 1)^{-1} c'$] we must solve the system of linear algebraic equations.

4. Numerical Results and Discussion

Eq. 17 gives the well-known eigenvalue problem for one-speed neutrons. It can be solved in a number of ways. Here, we have computed the critical radius R by the ordinary iterative procedure (S$\acute{e}$cant method). For this purpose, for given values of c and α we first compute the value of R. We than obtain the critical radius of the original problem from Eq. 4.

A computer program has been developed in C++ to perform this calculation. The results calculated in $F_1 - F_{12}$ approximations are given in Tables 1 and 2.

In Table 1 we compare our values for the isotropic results with other works for the different c values. The results agree within three or four digits with those calculated by different methods. Again we see that the result of Ref. [24] shows good agreement with our values. Clearly the method works equally well for large and small dimensions.

Table 2 shows the variation of the critical radius as a function of α. We see from Table 2 that the critical radius first increases with increasing anisotropy parameter α, but than decreases as α approaches 1/c. This means that the leakage is prevented by the forward anisotropy. Also, Table 2 contains the variational results of Pomraning [5] and Tezcan [6] for the value of c=1.1.

The agreement with our F_{12} = result is satifactory. For $\alpha \rightarrow 1/c$, this procedure appears to be adequate for our present purpose.

5. Conclusions

As pointed out before [5,6] we observe from the results that the critical radius varies nonmonotonically with anisotropy. That is, the forward scattering has an influence upon the critical radius.

Finally we note that, the F_N approximation consistently yields results accurate to three or four significant figures for the present consideration. It is a flexible and economical tool for solving transport problems in several fields. This fact makes the method of great practical interest.

6. Nomenclature

c = the smallest mean number of secondaries per collision for criticality

r = the distance measured in mean-free path units (mfp), from the axis of the cylinder.

θ = the angle between the z direction and unit vector Ω along the direction the of neutron.

ϕ = the angle between r and the projection of Ω on the xy plane.

α = real constant in the range of $0 < \alpha < 1$ and gives the fraction of particles which emerge from a collision in the forward direction.

Ω' = denotes the direction of the neutron velocity vector (and Ω after collision) and $\Omega' = \cos\theta = \mu$.

Σ_t = the total cross-section in the critical system.

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